International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 5 Issue 8 ǁ Aug. 2017 ǁ PP.75-79

A Remark on Approximating Permanents of Positive Definite Matrices

Ashutosh Mahapatra, Ranjana Pradhan, Sunita Priyadarsini, Bibekananda Biswal

Department of Mathematics, NM Institute of Engineering and Technology,Bhubaneswar , Odisha
Department of Mathematics, Raajdhani Engineering College,Bhubaneswar,Odisha
Department of Mathematics,Aryan Institute of Engineering and Technology Bhubnaeswar , Odisha
Department of Basic Science & Humanities ,Capital Engineering College,Bhubaneswar,Odisha

ABSTRACT: *Let A be an n n positive de nite Hermitian matrix with all eigenvalues between 1 and 2. We represent the permanent of A as the integral of some explicit log-concave function on R^{2n}. Consequently, there is a fully polynomial randomized approximation scheme (FPRAS) for per A.*

1991 Mathematics Subject Classi cation. 15A15, 15A57, 68W20, 60J22, 26B25.
***Key words** and phrases. permanent, positive de nite matrices, log-concave measures.*

INTRODUCTION AND MAIN RESULTS

Let A = (a_{ij}) be an n ,n complex matrix. The permanent of A is de ned as

n
X Y
per A = ^{a}k (k)$^{;}$
2S_n k=1

where S_n is the symmetric group of all n! permutations of the set f1; : : : ; ng. Re-cently, in particular because of connections with quantum optics, there was some interest in e cient computing (approximating) per A, when A is a positive semi-de nite Hermitian matrix, see [A+17], [GS18] and references therein. As is known, in that case per A is real and non-negative, see, for example, Chapter 2 of [Mi78]. In [A+17], Anari, Gurvits, Oveis Gharan and Saberi constructed a deterministic polynomial time algorithm approximating the permanent of a positive semide nite n n Hermitian matrix A within a multiplicative factor of c^n for c = e^{1+} 4:84, where 0:577 is the Euler constant. Similarly to the case of a non-negative real matrix A, the problem of exact computation of per A for a positive semide nite matrix A is #P-hard [GS18].

If A is a non-negative real matrix, a fully polynomial randomized approximation scheme (FPRAS) for per A was constructed by Jerrum, Sinclair and Vigoda [J+04]. Given an n n matrix non-negative A and a real 0 << 1, the algorithm of [J+04]

This research was partially supported by NSF Grant DMS 1855428. produces in (n=)$^{O(1)}$ time a number approximating per A within relative error . The algorithm is randomized, meaning that the number satis es the desired condition with a su ciently large probability p, for example, with p = 0:9 (then by running m independent copies of the algorithm and taking the median of the computed s, one can make the probability of error exponentially small in m). No such algorithm is known in the case of a positive semide nite Hermitian A, and the question of existence of an FPRAS in that case was asked in [A+17] and [GS18].

In this note, we show that that there is a fully polynomial randomized approx-imation scheme (FPRAS) for permanents of positive de nite matrices with the eigenvalues between 1 and 2. Namely, we represent per A for such an n n matrix A as the integral of an explicitly constructed log-concave function f_A : R^{2n} ! R_+, so that
Z
f_A(t) dt = per A:
$_R$2n

There is an FPRAS for integrating log-concave functions, see [LV07] for the detailed analysis and history of the Markov Chain Monte Carlo approach to the problem of integrating log-concave functions and a closely related problem of approximat-ing volumes of convex bodies. Hence the above integral representation

and an integration algorithm from [LV07] instantly produce an FPRAS for computing the permanent of a positive de nite Hermitian matrix with all eigenvalues between 1 and 2. We note that a standard interpolation argument implies that the problem of computing per A exactly remains #P-hard, when restricted to positive de nite matrices with eigenvalues between 1 and 2. Indeed, the set X_n of such n n ma-trices has a non-empty interior in the vector space of all n n Hermitian matrices. Given an arbitrary n n Hermitian matrix B, one can draw a line L through B and an interior point of X_n. Since the restriction of the permanent onto that line is a univariate polynomial of degree at most n, by computing the permanent per A_i for n + 1 distinct matrices A_i 2 (L \ X_n), we would be able to compute per B exactly by interpolation, which is a #P-hard problem, cf. [GS18].
We consider the space C^n with the standard norm

$kzk^2 = jz_1j^2 + : : : + jz_nj^2;$ where $z = (z_1; : : : ; z_n) :$

We identify $C^n = R^{2n}$ by identifying z = x + iy with (x; y). For a ~~com~~plex matrix

L = (l_{jk}), we denote by L = l_{jk} its conjugate, so that $l_{jk} = l_{kj}$ for all j; k:

We prove the following main result.
(1.1) Theorem. Let A be an n n positive de nite matrix with all eigenvalues between 1 and 2. Let us write A = I + B, where I is the n n identity matrix and B is an n n positive semide nite Hermitian matrix with eigenvalues between 0 2 and 1. Further, we write B = LL , where L = (l_{jk}) is an n n complex matrix. We de ne linear functions $‘_1; : : : ; ‘_n : C^n$! C by

$$‘_j(z) = \sum_{k=1}^{n} l_{jk} z_k \quad \text{for} \quad z = (z_1; : : : ; z_n) :$$

Let us de ne $f_A : C^n$! R_+ by

$$f_A(z) = \frac{1}{n} e \; zk^2 \prod_{j=1}^{n} 1 + j‘_j(z)j^2 \quad :$$

(1) Identifying $C^n = R^{2n}$, we have

$$\text{per } A = \int_{R^{2n}} f_A(x; y)\, dxdy:$$

(2) The function $f_A : R^{2n}$! R^{2n} and if

x = $x_1 + (1 \quad)x_2$ and

then
f_A (x; y)

R_+ is log-concave, that is, if $(x_1; y_1); (x_2; y_2)$ 2

y = $y_1 + (1 \quad)y_2$ for some 0 1

$f_A(x_1; y_1) f_A^{1} \quad (x_2; y_2)$:

2. Proofs

We start with a known integral representation of the permanent of a positive semide nite matrix.

(2.1) The integral formula. Let be the Gaussian probability measure in C^n with density

$^1{}_n$ e zk2 where $kzk^2 = jz_1j^2 + : : : + jz_nj^2$ for $z = (z_1; : : : ; z_n)$:

For the expectations of products of coordinates, we have

$$E\, z_i\, \overline{z_j} = \int_{C^n} z_i\, \overline{z_j}\, d\,(z) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i\ 6= j: \end{cases}$$

Let $`_1; : : : ; `_n : C^n$! C be linear functions and let $B = (b_{jk})$ be the n n matrix,

$$b_{jk} = E\, `_j \overline{`_k} = \int_{C^n} `_j(z)\overline{`_k(z)}\, d\,(z) \quad \text{for} \quad j; k = 1; : : : ; n:$$

Hence B is a positive semide nite Hermitian matrix and the Wick formula (see, for example, Section 3.1.4 of [Ba16]) implies that

$$(2.1.1) \quad \text{per}\, B = E\ j`_1j^2 \quad j`_nj^2 = \int_{C^n} j`_1(z)j^2 \quad j`_n(z)j^2\, d\,(z):$$

Next, we need a simple lemma.

(2.2) Lemma. Let $q : R^m$! R_+ be a positive semide nite quadratic form. Then the function

h(x) = ln 1 + q(x) q(x)

is concave.

Proof. It su ces to check that the restriction of h onto any a ne line x( ) = a+ b with a; b 2 R^m is concave. Thus we need to check that the univariate function

$G(\ ) = \ln 1 + (\quad + \quad)^2 + {}^2 \quad (\quad + \quad)^2 \quad {}^2$ for 2 R;

where 6= 0, is concave, for which it su ces to check that $G^{00}(\ )$ 0 for all . Via the a ne substitution := ( )= , it su ces to check that $g^{00}(\ )$ 0, where

$$g(\ ) = \ln 1 + {}^2 + {}^2 \qquad {}^2 {}_+ {}^2 {}_:$$

We have

$$g^0(\ ) = \frac{2}{1+{}^2+{}^2} \quad 2$$

and

$$g^{00}(\ ) = \frac{2(1+{}^2+{}^2) \qquad 4\,{}^2}{(1+{}^2+{}^2)^2} \quad 2$$

$$= \frac{2(1+{}^2+{}^2) \qquad 4\,{}^2 \quad {}_{21+}2_+2^2}{(1+{}^2+{}^2)^2}$$

$2+2^2+2^2 \ 4^2 \ 2\,2^4 \ 2^4 \ 4^2 \ 4^2 \ 4^{22}$

=

$(1+\ ^2+\ ^2)^2$

$6\ ^2$ $+2\ ^2+2\ ^4+2\ ^4+4\ ^{2\,2}$

= $(1+\ ^2+\ ^2)^2$ 0

and the proof follows.

(2.3) Proof of Theorem 1.1. We have

J f^X g

per A = per(I + B) = per B_J ;

1;::: ;n

where B_J is the principal jJj jJj submatrix of B with row and column indices in J and where we agree that per $B_;$ = 1. Let us consider the Gaussian probability measure in C^n with density$^n e^{zk2}$. By (2.1.1), we have

Y

per B_J = E $j`_j(z)j^2$

j2J

and hence

n Z

per A = E $1 + j`_j(z)j^2$ = $_{R2n}{}^{f}A^{(x;\ y)dxdy;}$

=1

j^Y

and the proof of Part (1) follows.

We write

n n

$_e$ zk^2 $1 + j`_j(z)j^2$ $_{=e}$ q(z) $1 + j`_j(z)j^2$ $_e$ $`_j\,(z)j^2$;

j^Y Y

=1 j=1

n

X_j

where q(z) = kzk^2 $j`_j(z)j^2$:

=1

By Lemma 2.2 each function $(1 + j`_j(z)j^2)e^{\ `j(z)j2}$ is log-concave on $R^{2n} = C^n$ and hence to complete the proof of Part (2) it su ces to show that q is a positive semide nite Hermitian form. To this end, we consider the Hermitian form

n n n n

p(z) = $j`_j(z)j^2$ = $^ljk^zk$ 2 = $^ljk_1{}^ljk_2{}^zk_1$ $\overline{^zk_2}$

j=1 j=1 k=1 j=1 $1k_1;k_2$ n

X X X X X

X
= ${}^{c}k_1k_2{}^{z}k_1{}^{z}k_2{}^{;}$

1 $k_1;k_2$ n

where

n
x_j

${}^{c}k_1k_2{}^{=l}jk_1{}^{l}jk_2$ for 1 k_1; k_2 n:
=1

Hence for the matrix C = (c_{k1k2}) of p, we have C = L L. We note that B = LL and that the eigenvalues of B lie between 0 and 1. Therefore, the eigenvalues of L L lie between 0 and 1 (in the generic case, when L is invertible, the matrices LL and L L are similar). Consequently, the eigenvalues of C lie between 0 and

1 and hence the Hermitian form q(z) with matrix I C is positive semide nite, which completes the proof of Part (2).

REFERENCES

[1]. [A+17] N. Anari, L. Gurvits, S. Oveis Gharan, and A. Saberi, Simply exponential approximation of the permanent of positive semide nite matrices, 58th Annual IEEE Symposium on Foundations of Computer Science { FOCS 2017, IEEE Computer Soc., Los Alamitos, CA, 2017, pp. 914{925.

[2]. [Ba16] A. Barvinok, Combinatorics and Complexity of Partition Functions, Algorithms and Combinatorics, 30, Springer, Cham, 2016.

[3]. [GS18] D. Grier and L. Schae er, New hardness results for the permanent using linear op-tics, Art. No. 19, 29 pp., 33rd Computational Complexity Conference, LIPIcs. Leibniz International Proceedings in Informatics 102, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018.

[4]. [J+04] M. Jerrum, A. Sinclair, E. Vigoda, A polynomial-time approximation algorithm for the permanent of a matrix with nonnegative entries, Journal of the ACM 51 (2004,), no. 4, 671{697.

[5]. [LV07] L. Lovasz and S. Vempala, The geometry of logconcave functions and sampling algo-rithms, Random Structures & Algorithms 30 (2007), no. 3, 307{358.

[6]. [Mi78] H. Minc, Permanents, Encyclopedia of Mathematics and its Applications, 6, Addison-Wesley Publishing Co., Reading, Mass., 1978.

[7]. Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA